THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 9 (April 8)

Squeeze Theorem. *Let $A \subseteq \mathbb{R}$, let $f, g, h \colon A \to \mathbb{R}$, and let c be a cluster point of A. If*

$$f(x) \leq g(x) \leq h(x) \quad \text{for all} \;\; x \in A, \; x \neq c,$$

and if $\lim_{x\to c} f = L = \lim_{x\to c} h$, then $\lim_{x\to c} g = L$.

Example 1. Deduce the following limits using the Squeeze Theorem.

(a) $\lim_{x\to 0} \cos x = 1$

(b) $\lim_{x\to 0} \left(\dfrac{\cos x - 1}{x} \right) = 0$

(c) $\lim_{x\to 0} \left(\dfrac{\sin x}{x} \right) = 1$

(d) $\lim_{x\to 0} (x \sin(1/x)) = 0.$

Definition. Let $A \subseteq \mathbb{R}$, let $f \colon A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Remarks. (1) If $c \in A$ is a cluster point of A, then f is continuous at c if and only if

$$f(c) = \lim_{x\to c} f(x).$$

(2) If $c \in A$ is not a cluster point of A, then f is automatically continuous at c since $A \cap V_\delta(c) = \{c\}$ for some $\delta > 0$.

Sequential Criterion for Continuity. *A function $f \colon A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

Example 2. Let $f \colon \mathbb{R} \to \mathbb{R}$ be the Dirichlet's function defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is not continuous at any point of $\mathbb{R}$.

Classwork

1. Determine the limit $\lim_{x\to 0} \frac{1}{x} \sin\left(\sin(x^2)\right)$.

   Solution. Note that $-|x| \le \sin x \le |x|$ for all $x \in \mathbb{R}$. Then

   $$-|x^2| \le -|\sin x^2| \le \sin(\sin x^2) \le |\sin x^2| \le |x^2| \quad \text{for all } x \in \mathbb{R}.$$

   Thus

   $$-|x| \le \frac{1}{x}\sin(\sin x^2) \le |x| \quad \text{for all } x \ne 0.$$

   Since $\lim_{x\to 0} |x| = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} \frac{1}{x} \sin\left(\sin(x^2)\right) = 0$. ◀

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Determine all the continuity point(s) of f. Justify your answer.

   Solution. The function f is continuous at 0 and is not continuous at any other points.

   Let $c = 0$. Given $\varepsilon > 0$, take $\delta = \varepsilon$. Now, if $x \in \mathbb{R}$ and $|x - c| < \delta$, then

   $$|f(x) - f(c)| = |f(x) - 0| = |f(x)| \le |x| < \delta = \varepsilon.$$

   Hence f is continuous at 0.

   Let $b \ne 0$. If $b \in \mathbb{Q}$, let (x_n) be a sequence in $\mathbb{R}\setminus\mathbb{Q}$ that converges to b. Then $f(x_n) = 0$ for all n, so that $\lim(f(x_n)) = 0$. However $f(b) = b \ne 0$.

   If $b \in \mathbb{R} \setminus \mathbb{Q}$, let (y_n) be a sequence in $\mathbb{Q}$ that converges to b. Then $f(y_n) = y_n \to b$. However $f(b) = 0$.

   In either case, f is not continuous at b. ◀